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# Spin Waves, Vortices, Fermions, and Duality in the Ising and Baxter Models

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Field-theoretic methods are applied to a number of two-dimensional lattice models with Abelian symmetry groups. It is shown, using a vortex + spin-wave decomposition, that the $Z_p$-Villain models are related to a class of continuum field theories with analogous duality properties. Fermion operators for these field theories are discussed. In the case of the Ising model, the vortices and spin-waves conspire to produce a free, massive Majorana field theory in the continuum limit. The continuum limit of the Baxter model is also studied, and the recent results of Kadanoff and Brown are rederived and extended.

## I. INTRODUCTION

There has been a substantial amount of progress recently in the analysis of two-dimensional lattice models with Abelian symmetry groups [1-5]. This class of models includes the planar model, the Ising model, and many others. The principal tool used is an extension of Kramers-Wannier-Kadanoff duality [6, 7] which catalogues excitations in terms of their vortex and spin-wave content. These methods have a natural extension to continuum field theories in two-dimensional space-time.

Most of the lattice models which have been studied are closely related to a field theory whose Euclidean Lagangian density can be written as

$$L = \frac{1}{2}(\nabla\phi)^2 + \frac{2y}{a^2}\cos(2J\tilde{\phi}) + \frac{2h_p}{a^2}\cos\left(\frac{p}{J}\phi\right). \tag{1.1}$$

The field $\phi$ is an order variable describing local fluctuations, while $\tilde{\phi}$ is the relatively non-local disorder variable dual to $\tilde{\phi}$. It can be defined in Minkowski space as

$$\tilde{\phi}(x,t) = \int_x^{\infty} dy\, \tfrac{1}{2}\dot{\phi}(y,t) - \int_{-\infty}^{x} dy\, \tfrac{1}{2}\dot{\phi}(y,t). \tag{1.2}$$

The parameter $a$ is the only length scale present, while $p$ is an integer. The dimensionless variables $y$ nd $h_p$ are the activities of vortices and spin-waves, respectively.

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If $y$ and $h_p$ are both set equal to zero, the free field theory of a massless scalar field is obtained. This is the continuum limit of the Gaussian model, and it is reviewed in Appendix A. If $h_p = 0$, then $L$ is essentially the Lagrangian of the sine-Gordon model. This is closely related to the planar model. A non-zero value of $h_p$ means that a $p$-fold symmetry-breaking external field has been applied. For example, $p = 1$ is analogous to an external magnetic applied to a spin system.

In this paper I will primarily be concerned with the field theories associated with the $Z_p^v$ models, the Ising model and the Baxter model. The $Z_p^v$ models are obtained from the better known $Z_p$ models by applying the Villain approximation [2–4]. In the next Section, I show how field-theoretic methods can be applied to the analysis of these models. In particular, I discuss their duality properties, demonstrating that fractional vortex operators are needed to implement duality in correlation functions. I also define fermion operators for these models. The $Z_2^v$ model is better known as the Ising model, and it is Studied in Section III. The results of Section II are used to show how the vortex and spin-wave operators combine to produce the field theory of a free, massive Majorana fermion in the continuum limit [8, 9]. Section IV studies the continuum field theory associated with the Baxter and Ashkin–Teller models. The recent results of Kadanoff and Brown [5] are rederived and extended.

## II. Properties of the $Z_p^v$ Models

This section deals with the $Z_p^v$ models in two dimensions. These models arise naturally as the Villain approximation to the more familiar $Z_p$ models, and are closely related to the planar model. I begin by showing how the $Z_p^v$ models arise naturally from the $Z_p$ models. Following the work of Kadanoff [2, 4] and Elitzur *et al.* [3], the $Z_p^v$ models are given a simple form in which the topological structure of the relevant excitations is manifest. I then write down a Euclidean field theory with the same structural properties. From this, I deduce duality relations and write down for future use a formula for the fermion operators associated with the models.

The $Z_p$ models are lattice models which have a classical two-component spin of unit length at each lattice site $i$, described by angle $\phi_i$. The $Z_p$ models differ from the planar model in that $\phi_i$ can take on only $p$ different values:

$$\phi_i = 2\pi n_i/p, \tag{2.1}$$

where $n_i = 1,..., p$. Formally, the $Z_p$ models can be obtained by placing the planar model in infinitely strong $p$-fold symmetry-breaking fields. The $Z_2$ model is better known as the Ising model.

The Euclidean action for the $Z_p$ model is given by a conventional nearest-neighbor interaction:

$$A_1 = J_1^2 \sum_{\langle ij \rangle} \left\{ 1 - \cos\left[ \frac{2\pi}{p} (n_i - n_j) \right] \right\}. \tag{2.2}$$

The partition function $Q_p$ is a function of the dimensionless coupling constant $J_1$ and is given by

$$Q_p = N_1 \sum_{\{n_j\}} \exp - A_1[n_j], \tag{2.3}$$

where the sum is over all sets of integers between 1 and $p$, and $N_1$ is a normalization factor.

The $Z_p^v$ models are obtained by applying the Villain approximation [26] to $Q_p$. In this case, the approximation is

$$\exp \left\{ J_1^2 \left[ \cos \left( \frac{2\pi n}{p} \right) - 1 \right] \right\} \simeq A \sum_l \exp \left\{ \frac{-1}{2} J_2^2 \left( \frac{2\pi n}{p} - 2\pi l \right)^2 \right\}, \tag{2.4}$$

where $J_2$ and $A$ are functions of $J_1$ such that $J_2 \to J_1$ and $A \to 1$ as $J_1 \to \infty$. For the case $p = 2$, the approximation can be made exact by choosing $A$ and $J_2$ to satisfy

$$A \sum_l \exp \left\{ \frac{-1}{2} J_2^2 (2\pi l)^2 \right\} = 1, \tag{2.5}$$

$$A \sum_l \exp \left\{ \frac{-1}{2} J_2^2 (2\pi l + \pi)^2 \right\} = e^{-2J_1^2}. \tag{2.6}$$

In general, the $Z_p^v$ model is not the same as the corresponding $Z_p$ model. For example, the $Z_4$ model is equivalent to two decoupled Ising models, but the $Z_4^v$ model is a particular case of the Ashkin–Teller model [4].

I can now obtain the $Z_p^v$ models as a approximation to the $Z_p$ models. At every nearest-neighbor interaction, the Villain approximation is made, introducing integer bond variables $s_{ij}$. The factors of $A$ can be absorbed into a new normalization constant $N_2$. Thus, the partition $Q_p^v$ of the $Z_p^v$ model is defined by

$$Q_p^v = N_2 \sum_{\{s_{ij}\}} \sum_{\{n_i\}} \exp - A_2[n_i s_{ij}], \tag{2.7}$$

where the Euclidean action is given by

$$A_2 = \frac{1}{2} J_2^2 \sum_{\langle ij \rangle} \left[ \frac{2\pi}{p} (n_i - n_j) - 2\pi s_{ij} \right]^2. \tag{2.8}$$

The Poisson summation formula can now be used to eliminate the orginal $n_i$ variables. This formula can be written as

$$\sum_n \delta \left( \phi - \frac{2\pi n}{p} \right) = \frac{p}{2\pi} \sum_m e^{ipm\phi}. \tag{2.9}$$

From this, it follows that

$$\sum_{n=1}^{p} f(2\pi n/p) = (p/2\pi) \int_0^{2\pi} d\phi \sum_m e^{ipm\phi} f(\phi). \tag{2.10}$$

At each lattice site $i$ the discrete variable $n_i$ is replaced by new variables $m_i$ and $\phi_i$; $m_i$ is an integer, while $\phi_i$ takes on all values between 0 and $2\pi$. The new form for $Q_p^v$ is

$$Q_p^v = N_3 \sum_{\{m_i, s_{ij}\}} \int_0^{2\pi} \prod_i d\phi_i \exp - A_3[\phi_i, m_i, s_{ij}], \tag{2.11}$$

where the Euclidean action $A_3$ has the form

$$A_3 = \sum_{\langle ij \rangle} \tfrac{1}{2} J_2^2 [\phi_i - \phi_j - 2\pi s_{ij}]^2 + \sum_j ipm_j \phi_j. \tag{2.12}$$

I now argue that it does no harm to allow the $\phi_i$ integrations to run from $-\infty$ to $+\infty$. Suppose I let some particular $\phi_j$ to run from $-2\pi L$ to $2\pi L$. This multiplies $Q_p$ by a factor of $2L$; it does not change correlation functions at all. The same change in range of integration can be made for all $\phi_j$'s, and then the limit $L \to \infty$ can be taken; $N_3$ can swallow the infinity. Therefore, $Q_p^v$ is just

$$Q_p^v = N_3 \sum_{\{m_i, s_{ij}\}} \int_{-\infty}^{+\infty} \prod_i d\phi_i \exp - A_3[\phi_i, m_i, s_{ij}]. \tag{2.13}$$

If all the $m_j$'s are set equal to zero, Eq. (2.13) is the Villain approximation to the planar model. The role of the $m_j$'s is to explicity break the planar model symmetry. As in the planar model, one can distinguish between vortex and spin-wave excitations. A non-zero $m_j$ behaves as a source for a spin-wave excitation at the lattice site $j$. Non-zero $s_{ij}$'s give rise to vortex excitations, which live on the dual lattice.

If all the $s_{ij}$'s are set equal to zero, still another representation of the Villain approximation to the planar model is obtained [1]:

$$A_3 = \sum_{\langle ij \rangle} \frac{+1}{2} J_2^2 [\phi_i - \phi_j]^2 + \sum_j ipm_j \phi_j. \tag{2.14}$$

This is a manifestation of the self-duality of the $Z_p^v$ models in two dimensions. As a consequence, the vortex and spin-wave activities are equal. This has been proven by Kadanoff [2] and Elitzur *et al.* [3].

Armed with this information, I can now write down a Euclidean Lagrangian density $L_p$ which defines an ultraviolet-regulated continuum field theory with the same structure as the $Z_p^v$ model. It is given by

$$L_p = \frac{1}{2}(\nabla\phi)^2 + \sum_{m=1}^{\infty} \frac{2(y)^{m^2}}{2^2} \cos\left(\frac{pm}{J}\phi\right) + \sum_{n=1}^{\infty} \frac{2(y)^{n^2}}{a^2} \cos(2\pi nJ\tilde{\phi}). \quad (2.15)$$

The dimensionless coupling constant $J$ equals $J_2$, while $a$ is an ultraviolet cutoff with dimensions of length. The dimensionless number $y$ can be considered to be the activity of both spin-waves and vortices, but it is not a free parameter of the theory. It parametrizes the relation of the bare mass in the Lagrangian to the cutoff $a^{-1}$, and is a function of $J$. It is also extremely useful as a bookkeeping device. The non-local field $\tilde{\phi}$ is defined in Minkowski space in the usual way:

$$\tilde{\phi}(x, t) = \int_x^{\infty} dy\, \tfrac{1}{2}\dot{\phi}(y, t) - \int_{-\infty}^{x} dy\, \tfrac{1}{2}\dot{\phi}(y, t). \quad (2.16)$$

If I define a functional integral $R_p$ as

$$R_p = \int [d\phi] \exp\left[-\int d^2x L_p(\phi)\right] \quad (2.17)$$

and expand in a power series about $y = 0$, I obtain an expression for $R_p$ very much like that for $Q_p^v$. The $\cos((pm/J)\tilde{\phi})$ terms behave like sources for spin-wave excitations, while the $\cos(2\pi nJ\phi)$ terms act as sources for vortices. This expansion is discussed in Appendix B. $Q_p^v$ and $R_p$ differ is only two ways: In the continuum theory, the excitations are not restricted to discrete locations, and the Laplacian replaces the corresponding lattice operator. As in the case of the planar model and the sine-Gordon theory, the critical properties of the two models are the same.

It is straightforward to check that $L_p$ has the same properties as the $Z_p^v$ models. To lowest order in $y$, the spin-wave operator $\cos(p\phi/J)$ will be relevant if

$$(p/J)^2 < 8\pi, \quad (2.18)$$

while the vortex operator $\cos(2\pi J\tilde{\phi})$ is relevant for $J$ in the range

$$(2\pi J)^2 < 8\pi. \quad (2.19)$$

For $p \geqslant 5$, there will be an intermediate region for which both operators are irrelevant. To zeroth order in $y$, this range is

$$\frac{p^2}{8\pi} > J^2 > \frac{2}{\pi} \quad (2.20)$$

In this region, the $Z_p^v$ models have the infrared behavior of the Gaussian model. It can be shown that the $L_p$ models lead to the same renormalization group equations as the corresponding lattice models.

As usual, it is possible to define spin-wave and vortex operators. Spinwave operators $S_m$ are given by

$$S_m = \exp \frac{im}{J} \phi. \tag{2.21}$$

Vortex operators are defined as

$$V_n = \exp[i2\pi nJ\tilde{\phi}] \tag{2.22}$$

It is easy to check that this field theory has the same duality properties as $Z_p^v$. This follows from the duality properties of the free massless scalar field, as discussed in Appendix B. Order by order in $y$, there is manifest duality between $\phi$ and $\tilde{\phi}$, so that the generating functional $R_p$ is invariant under the change

$$2\pi J \leftrightarrow p/J. \tag{2.23}$$

This duality extends to correlation functions if a fractional vortex operator is introduced. I define

$$V_{n/p} = \exp\left[i\frac{2\pi nJ}{p}\tilde{\phi}\right] \tag{2.24}$$

so that the correlation functions are invariant under the duality transformation

$$2\pi J/p \leftrightarrow 1/J, \tag{2.25}$$

$$S_m \leftrightarrow V_{m/p}. \tag{2.26}$$

It is interesting to note that the $S_m$'s and $V_{n/p}$'s obey a variant of t'Hooft's order–disorder commutation relations [10]. The contraction of a spin-wave operator $Q_m(x)$ with a vortex operator $V_{n/p}(y)$ gives rise to a term

$$\exp\left[-i\frac{nm}{p}\theta(x-y)\right], \tag{2.27}$$

where $\theta(x-y)$ is the usual arctangent function, that measures the angle $(x-y)$ makes with some fixed axis. It is easy to see that

$$S_m(x)\, V_{n/p}(y) = \exp\left[\pm i\frac{nm\pi}{p}\right] V_{n/p}(y)\, S_m(x). \tag{2.28}$$

The sign ambiguity is due to the cut in the arctangent function. This relation can also be derived at equal times in Minkowski space, using the canonical commutation relation and definition (2.24). Equation (2.28) is not surprising, given the interpretation of the spin-waves and vortices as electric and magnetic charges [2].

It will be convenient in the next section to have available the fermion operators introduced by Mandelstam [11]. As discussed, for example, by Swieca [12], a whole class of fermion operators can be made from $\phi$ and $\tilde{\phi}$, defined by

$$\psi = \exp\left[i\frac{\beta}{2}\gamma_5\phi + i\frac{2\pi}{\beta}\tilde{\phi}\right]U_0, \tag{2.29}$$

where $\beta$ is arbitrary and $U_0$ is a constant spinor of the form

$$U_0 = \left(\frac{M}{2}\right)^{1/2}\begin{pmatrix}1\\1\end{pmatrix}. \tag{2.30}$$

The constant $M$ has dimensions of mass. In this case, the logical choice is

$$\beta = p/J \tag{2.31}$$

by analogy with the sine-Gordon model.

## III. The Ising Model

I will now apply the results of the previous section to the two-dimension Ising model. In this case, the Lagrangian is $L_2$, given by

$$L_2 = \frac{1}{2}(\nabla\phi)^2 + \frac{2y}{a^2}\cos\left(\frac{2}{J}\phi\right) \pm \frac{2y}{a^2}\cos(2\pi J\tilde{\phi}). \tag{3.1}$$

In writing (3.1), I have made a few changes from Eq. (2.15). First I have dropped all those cosine operators which are irrelevant near the critical point. Secondly, I have noted explicitly the inherent ambiguity in the relative sign of the vortex and spin-wave operators.

The field theory described by (3.1) is not the continuum limit of the Ising model, which is obtained by taking $a$ to zero while keeping the physical mass fixed. The Lagrangian $L_2$ is associated with an ultraviolet-regulated field theory designed to reproduce the relevant topological features of the Ising model, including the duality between order and disorder operators. It is, therefore, somewhat more phenomenological in nature rather than fundamental.

There are a number of approaches which can be used to analyze the model described by (3.1). One method is to generalize $L_2$ to

$$L_2' = \frac{1}{2}(\nabla\phi)^2 + \frac{2h_2}{a^2}\cos\left(\frac{2}{J}\phi\right) \pm \frac{2y}{a^2}\cos(2\pi J\tilde{\phi}). \tag{3.2}$$

This Lagrangian describes the planar model in a two-fold symmetry breaking field. It is expected to lie in the same universality class as the two-dimensional Ising model.

José *et al.* [1] have developed renormalization group equations for this model for $y$ and $h_2$ small. These and similar equations have proven very useful in the study of the Baxter and Ashkin–Teller models [4, 5].

The approach I will follow in this section is more modest. The generating functional $Z$ associated with $L_2$ is essentially the partition function of the Ising model. By manipulating $L_2$, I will show that in the continuum limit $Z$ is precisely the generating functional of a Majorana fermion field theory, in agreement with the work of Schultz *et al.* [8].

As usual, duality locates the critical point exactly. It is obtained from Eq. (2.23), which locates the critical $J_c$ at

$$J_c = 1/\sqrt{\pi}. \tag{3.3}$$

At this point, $L_2$ is manifestly self-dual. The conventional critical value of $J_1$ is given by

$$e^{-2J_1^2} = \sqrt{2} - 1. \tag{3.4}$$

To see that (3.3) and (3.4) are equivalent, note that $J_1$ and $J_2$ are related by (2.5) and (2.6), which state that

$$e^{-2J_1^2} = \sqrt{k} \tag{3.5}$$

when

$$J_2^2 = \frac{1}{2\pi} \frac{K'(k)}{K(k)}, \tag{3.6}$$

where $K$ and $K'$ are the usual complete elliptic integrals. Use of the exact relation

$$\frac{K'(3-2\sqrt{2})}{K(3-2\sqrt{2})} = 2 \tag{3.7}$$

shows that (3.4) implies

$$J_2^2 = 1/\pi. \tag{3.8}$$

I will now take the cutoff $a$ to zero, holding the physical masses fixed. On dimensional grounds, a physical mass $M_p$ must be related to $a$ and $J$ by

$$M_p = a^{-1} F(J). \tag{3.9}$$

Therefore, $J$ must be taken to $J_c$ as $a$ goes to zero. The resulting field theory will have a single free parameter, a mass which sets the scale of the theory. It is easy to see from $L_2$ what the Lagrangian density $I_c$ of this theory must be; it is given by

$$L_c = \tfrac{1}{2}(\nabla\phi)^2 + m^2 \cos\sqrt{4\pi}\,\phi \pm m^2 \cos\sqrt{4\pi}\,\tilde{\phi}, \tag{3.10}$$

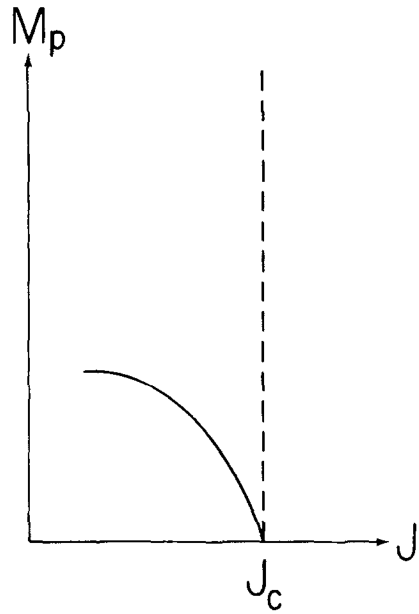


FIG. 1. The physical mass $M_p$ as a function of $J$ for fixed $a$. The dashed line represents the scaling limit $J = J_c$, $a = 0$, $M_p$ finite.

All necessary renormalizations are understood, although I have not indicated them there.

The significance of this limiting theory is easy to explain. It should be, and is, identical to the continuum or scaling limit of the two-dimensional Ising model. In Fig. 1, I have plotted $M_p$ versus $J$ for constant $a$. The dotted line at $J = J_c$ represents the scaling limit which has $a = 0$ but $M_p$ finite. This field theory is Euclidean-invariant at all length scales. The point $J = J_c$, $M_p = 0$ represents a massless field theory. There are two different ways this point can be approached: one can take $a$ to zero and then $M_p$, or vice versa. The fundamental hypothesis of the field-theoretic approach to critical phenomena is that these two limits commute, giving rise to the same correlation functions and critical indices. Just as the original lattice theory had one relevant parameter, $J_1$, the scaling limit theory will also have one relevant parameter, the physical mass $M_p$.

Equation (3.10) is less than transparent. It becomes much more meaningful when $L_c$ is written in terms of fermion fields. Care must be exercised, for some of the usual equivalences [13] do not hold. For example, the curl of $\phi$, $\varepsilon_{\mu\nu}\partial^\nu\phi$, us no longer conserved, due to the cos $\sqrt{4\pi}\,\tilde{\phi}$ term in $L_s$. Therefore, the curl of $\phi$ cannot be identified with a conserved fermion current. However, the relation between the $\phi$, $\tilde{\phi}$, and the fermion operator $\psi$, Eq. (2.29), still holds, at least order by order in $M^2$. As usual, the cos $\sqrt{4\pi}\,\phi$ term is equivalent to a fermion mass term

$$m^2 \cos \sqrt{4\pi}\,\phi = M\bar{\psi}\psi, \tag{3.11}$$

where the mass $M$ has replaced $m^2$ as the free parameter. The cos $\sqrt{4\pi}\,\tilde{\phi}$ term is less familiar, but Eq. (2.29) can be used to show that

$$m^2 \cos \sqrt{4\pi}\,\tilde{\phi} = M(\psi_1 \psi_2 + \psi_2^+ \psi_1^+). \tag{3.12}$$

The relative sign of $\psi_1 \psi_2$ and $\psi_2^+ \psi_1^+$ is fixed by hermiticity. Putting these two equivalences together, I have

$$m^2[\cos \sqrt{4\pi}\,\phi \pm \cos \sqrt{4\pi}\,\tilde{\phi}] = M[\psi_1^+ \psi_2 + \psi_2^+ \psi_1 \pm \psi_1 \psi_2 \pm \psi_2^+ \psi_1^+]. \tag{3.13}$$

Equation (3.13) is nothing but a mass term for a Majorana fermion. To see why this is so, consider an ordinary Dirac fermion theory. The Lagrangian density $L_{\mathrm{D}}$ is

$$L_{\mathrm{D}} = \bar{\psi} i\gamma \cdot \partial\psi - M\bar{\psi}\psi. \tag{3.14}$$

In the $\gamma$-matrix representation being used [14], the $\gamma$-matrices are real. As a consequence, charge conjugation is given by

$$\psi \to \psi_{\mathrm{c}} = \gamma_5 \psi^+. \tag{3.15}$$

Using this transformation rule, I can write down Majorana fermion operators which are self-conjugate; they can be defined as

$$\psi_A \equiv \frac{1}{\sqrt{2}} [\psi + \gamma_5 \psi^+], \tag{3.16}$$

$$\psi_B \equiv \frac{1}{\sqrt{2}} [\psi - \gamma_5 \psi^+]. \tag{3.17}$$

It is easy to show that

$$\bar{\psi}_{A,B} \psi_{A,B} = \bar{\psi}\psi \pm (\psi_1^+ \psi_2^+ + \psi_2 \psi_1), \tag{3.18}$$

so that Eq. (3.13) describes a Majorana mass term. Therefore, I can write $L_s$ as

$$L_s = \tfrac{1}{2} \bar{\psi}_A i\gamma \cdot \partial\psi_A - \tfrac{1}{2} M \bar{\psi}_A \psi_A. \tag{3.19}$$

Of course, $\psi_B$ could have been used instead of $\psi_A$. The unfamiliar factor of $\frac{1}{2}$ occurs because $\bar{\psi}_A$ is not independent of $\psi_A$; I have rescaled $M$ to accommodate this factor.

Equation (3.19) is the central result of this section, but it is hardly new. It was shown in 1964 by Schultz *et al.* [8] that the two-dimensional Ising model can be regarded as a lattice field theory of a free Majorana fermion. Since that time, this aspect of the model has been used extensively, culminating in the work of Sato *et al.* [15], who derived the Ising model correlation functions in the continuum limit using the Majorana fermion representation. What is surprising is that the Majorana field theory falls out of the spin-wave–vortex formalism so easily.

## IV. The Baxter and Ashkin–Teller Models

The Baxter model [16], also known as the eight-vertex model, can be written as two interacting Ising models [17]. Consider the unit cell shown in Fig. 2. The spins $\sigma_1$ and $\sigma_3$ are coupled in the usual way, as are $\sigma_2$ and $\sigma_4$. In addition, there is a four-spin coupling. Thus, the contribution of this cell to the reduced Hamiltonian can be written as

$$-K_1\sigma_1\sigma_3 - K_2\sigma_2\sigma_4 - \lambda\sigma_1\sigma_2\sigma_3\sigma_4. \tag{4.1}$$

The free energy of this model was determined by Baxter [16]; from this he was able to show that the critical exponent $\alpha$ is a continuous function of $\lambda$.

A closely related model is the Ashkin–Teller model [18], which can also be understood using Ising models [19]. At every lattice site $i$, there are two Ising spins $\sigma_i^{(1)}$ and $\sigma_i^{(2)}$. The reduced Hamiltonian is given by a sum over nearest-neighbor pairs:

$$H_{\mathrm{AT}} = \sum_{\langle ij\rangle} [-K_1\sigma_1^{(1)}\sigma_j^{(1)} - K_2\sigma_i^{(2)}\sigma_j^{(2)} - K_4\sigma_i^{(1)}\sigma_j^{(1)}\sigma_i^{(2)}\sigma_j^{(2)}]. \tag{4.2}$$

Like the Baxter model, the Ashkin–Teller model reduces to two decoupled Ising models when the four-spin coupling ($K_4$) is zero. The critical indices of this model are also continuous functions of $K_4$. Not surprisingly, the Baxter model and the Ashkin–Teller model are dual to one another [20].

A great deal is known about the field theory associated with the continuum limit of two decoupled Ising models whose nearest-neighbor couplings, $K_1 K_2$, are equal [21]. Given the results of the previous section, it is easy to see why this is so. Two free Majorana fermions can be used to form one Dirac fermion. Thus, the field theory associated with the doubled Ising model is that of a free, massive Dirac fermion:

$$L_{\mathrm{D}} = \bar{\psi}(i\gamma\cdot\partial - M)\psi \tag{4.3}$$

The order and disorder variables survive the continuum limit. I denote the two order fields as $\sigma^{(1)}(x)$ and $\sigma^{(2)}(x)$, while the dual disorder fields are $\mu^{(1)}(x)$ and $\mu^{(2)}(x)$. Certain composite fields have very simple representations. In particular, $\sigma^{(1)}\sigma^{(2)}$ and $\mu^{(1)}\mu^{(2)}$ are given by

$$\sigma_{\mathrm{D}} \equiv \sigma^{(1)}\sigma^{(2)} = \sin\sqrt{\pi}\,\phi, \tag{4.4a}$$

$$\mu_{\mathrm{D}} \equiv \mu^{(1)}\mu^{(2)} = \cos\sqrt{\pi}\phi, \tag{4.4b}$$

where $\phi$ is the scalar field associated with the free Dirac field theory in the usual way [13, 21]. Duality is implemented in this field theory by a $\gamma_5$ transformation on $\psi$, $\psi\to\gamma_5\psi$, which changes the sign of the mass. Under this duality transformation, $\sigma_{\mathrm{D}}$ and $\mu_{\mathrm{D}}$ transform as

$$\sigma_{\mathrm{D}} \to \mu_{\mathrm{D}}, \tag{4.5a}$$

$$\mu_{\mathrm{D}} \to -\sigma_{\mathrm{D}}. \tag{4.5b}$$

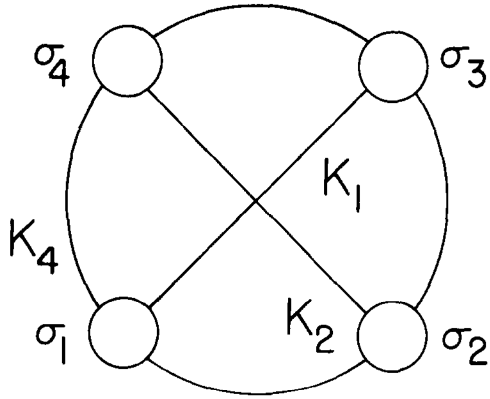


FIG. 2. The Baxter model couplings. $K_1$ and $K_2$ are next-nearest-neighbor couplings, while $K_4$ couples the four spins of a unit cell.

Given the success of this approach, I am going to write down a field theory Lagrangian which should describe the general continuum limit of the Baxter and Ashkin–Teller models. It is given by

$$L = \bar{\psi}_A(i\gamma \cdot \partial - M_A)\,\psi_A + \bar{\psi}_B(i\gamma \cdot \partial - M_B)\,\psi_B + (g/2)(\bar{\psi}_B\psi_B). \tag{4.6}$$

This can be proven for two special cases. When $g$ is set equal to zero, $L$ reduces to two free Majorana field theories, associated with two noninteracting Ising Models. If $M_A = M_B$, $L$ can be identified with the continuum limit of the Baxter model in the case $K_1 = K_2$ [22]. Even when $M_A \neq M_B$, $L$ is formally equivalent to the general Baxter model; this can be shown using the arguments of Berg [23]. One simply shows that $(\bar{\psi}_A\psi_A)(\bar{\psi}_B\psi_B)$ is essentially a four-spin coupling. A final justification comes from exhaustion: $L$ is the most general interacting local Lagrangian that can be made from two Majorana fields.

Consider now the special case of $L$ given by the restriction $M_A = M_B$. In that case $L$ is equivalent to the Massive Thirring model Lagrangian:

$$L = \bar{\psi}(i\gamma \cdot \partial - M)\psi - \tfrac{1}{2} g j_\mu j_\mu. \tag{4.7}$$

In turn, this is equivalent to the sine-Gordon theory [13], with Lagrangian

$$L = \frac{1}{2}(\partial\phi)^2 + \frac{\alpha}{\beta^2}\cos\beta\phi. \tag{4.8}$$

The coupling constants $\beta$ and $g$ are related by

$$1 + \frac{g}{\pi} = \frac{4\pi}{\beta^2}. \tag{4.9}$$

It is easy to show that $\beta$ is related to the coupling constant $J$ used in the previous section by

$$\beta = 2/J. \tag{4.10}$$

Luther and Peschel [22] have shown that $\sigma_{\mathrm{D}}$ can be identified for arbitrary $\beta$:

$$\sigma_{\mathrm{D}} = \cos\left(\frac{\beta}{2}\phi\right). \tag{4.11}$$

Because duality is implemented by a $\gamma_5$ transformation on the fermion fields, the associated boson field transforms as

$$\phi \to \phi + \pi/\beta. \tag{4.12}$$

It follows that the disorder field $\mu_{\mathrm{D}}$ dual to $\sigma_{\mathrm{D}}$ must be given by

$$\mu_{\mathrm{D}} = \sin\left(\frac{\beta}{2}\phi\right). \tag{4.13}$$

It is also interesting to find the continuum limit $E$ of the energy density $E_i$. It is defined as

$$E_i = E_i^{(1)} + E_i^{(2)} = \sum_{\text{n.n.}} \sigma_i^{(1)}\sigma_j^{(1)} + \sigma_i^{(2)}\sigma_j^{(2)}, \tag{4.14}$$

where the sum is over the nearest neighbors of $i$. The fact that $M$ is proportional to $T - T_c$ implies $E$ is essentially $\bar{\psi}\psi$, or

$$E \sim \cos(\beta\phi). \tag{4.15}$$

Note that the identification of the sine-Gordon theory with the Baxter model breaks down when $\beta^2$ reaches $8\pi$, the point at which $\bar{\psi}\psi$ becomes marginal.

Further progress is made possible by considering a dual transformation on $\psi_B$ only. This corresponds to making a dual transformation on one Ising sublattice, which turns the Baxter model into the Ashkin–Teller model. This dual transformation has the effect of flipping the signs of $M_B$ and $g$. It is not hard to show, using (2.29), that this model is equivalent to a sine-Gordon theory with Lagrangian

$$L' = \frac{1}{2}(\partial\tilde{\phi})^2 + \frac{\alpha}{\beta^2}\cos\left(\frac{4\pi}{\beta}\tilde{\phi}\right). \tag{4.16}$$

Note that in the field theory of the Ashkin–Teller model, it is $\bar{\psi}_A\psi_A - \bar{\psi}_B\psi_B$ that is relevant. It becomes marginal when $\beta^2 = 2\pi$, and is irrelevant for $\beta^2 < 2\pi$.

The dual transformation replaces $\sigma^{(2)}$ by $\mu^{(2)}$, but leaves $\sigma^{(1)}$ alone. Therefore, it is $\sigma^{(1)}\mu^{(2)}$ that now have simple representations as functions of $\tilde{\phi}$:

$$\sigma^{(1)}\mu^{(2)} = \cos\left(\frac{2\pi}{\beta}\tilde{\phi}\right), \tag{4.17}$$

$$\mu^{(1)}\sigma^{(2)} = \sin\left(\frac{2\pi}{\beta}\tilde{\phi}\right). \tag{4.18}$$

The operator which is multiplied by $T - T_c$ in $L'$ is not $E^{(1)} + E^{(2)}$, but

$$\tilde{E} = E^{(1)} - E^{(2)} = \cos\left(\frac{4\pi}{\beta}\tilde{\phi}\right). \tag{4.19}$$

I can now summarize these identifications in the form of a dictionary, Table I, which gives the continuum limit of lattice observables as boson operators. These identifications are consistent with the work of Kadanoff and Brown, who used operator product expansions to study the critical line $M = 0$ around $\beta^2 = 4\pi$. What has been gained here is that these results are now proved valid for $M \neq 0$ as well. For the convenience of the reader, I list in Table II the equations which relate $\beta$, $g$, and $J$, and also the parameter $K$ used by Kadanoff and Brown [5].

One final question must be answered: What is the relation between the Baxter model four-spin coupling $\lambda$ and the parameters $g$ and $\beta$? A relation was derived by Kadanoff and Brown using the operator product expansion at the decoupling point

TABLE I

| Sine-Gordon operator $O$ | Baxter operator | $\eta_0$[a] |
|---|---|---|
| $\sin\frac{\beta}{2}\phi$ | $\sigma_D \equiv \sigma^{(1)}\sigma^{(2)}$ | $\frac{\beta^2}{8\pi}$ |
| $\cos\frac{\beta}{2}\phi$ | $\mu_D \equiv \mu^{(1)}\mu^{(2)}$ | $\frac{\beta^2}{8\pi}$ |
| $\cos\frac{2\pi}{\beta}\tilde{\phi}$ | $\sigma^{(1)}\mu^{(2)}$ | $\frac{2\pi}{\beta^2}$ |
| $\sin\frac{2\pi}{\beta}\phi$ | $\sigma^{(2)}\mu^{(1)}$ | $\frac{2\pi}{\beta^2}$ |
| $\cos(\beta\phi)$ | $E \equiv E^{(1)} + E^{(2)}$ | $\frac{\beta^2}{2\pi}$ |
| $\sin\frac{4\pi}{\beta}\tilde{\phi}$ | $E \equiv E^{(1)} - E^{(2)}$ | $\frac{8\pi}{\beta^2}$ |

[a] $\eta_0$ is defined by the $\alpha = 0$ theory: $\lim_{|x-y|\to\infty}\langle O(x)\,O(y)\rangle \sim |x-y|^{-\eta_0}$.

TABLE II

| |
|---|
| $\frac{\beta^2}{4\pi} = \left(1 + \frac{g}{\pi}\right)^{-1}$ |
| $g = \pi\left(\frac{4\pi}{\beta^2} - 1\right)$ |
| $\beta = \frac{2}{J}$ |
| $g = \pi(\pi J^2 - 1)$ |
| $\beta^2 = \frac{4}{K}$ |
| $J^2 = K$ |

$M = 0$, $g = 0$ [5]. It turns out that a direct proof can be given which is valid for arbitrary $M$ and $g > (-\pi/2)$. The essence of the proof is contained in the work of Luther and Peschel [22]. It rests on a series of equivalences, which are

Baxter Model = $XYZ$ Spin Chain
= Lattice Massive Thirring Model
→ Massive Thirring Model
(Continuum Limit)
= Sine-Gordon Theory.

The first equivalence was proven by Sutherland [24], who showed that with an appropriate choice of parameters, the Hamiltonian of the $XYZ$ spin chain commutes with the transfer matrix of the Baxter model. The Hamiltonian of the $XYZ$ spin chain is given by

$$H_{XYZ} = -\sum_i J_x S_i^x S_{i+1}^x + J_y S_i^y S_{i+1}^y + J_z S_i^z S_{i+1}^z. \tag{4.20}$$

The correct choice of parameters $J_x$, $J_y$, and $J_z$ is

$$J_x = 1, \tag{4.21a}$$

$$J_y = \sinh 2K_1 \sinh 2K_2 + \cosh 2K_1 \cosh 2K_2 \tanh 2\lambda, \tag{4.21b}$$

$$J_z = \tanh 2\lambda. \tag{4.21c}$$

After a Jordan–Wigner transformation, the $XYZ$ spin chain becomes a lattice version of the Massive Thirring model.

A continuum theory is obtained from the lattice theory by taking $J_y$ to 1 with physical masses fixed. The correct relationship between $J_z$ and $g$ was given by Lüscher [25], who showed

$$J_z = (-1)\cos\pi\frac{(\pi+2g)}{(2\pi+2g)}. \tag{4.22}$$

Using Eq. (4.9), this is

$$J_z = (-1)\cos\left[\pi\left(1-\frac{\beta^2}{8\pi}\right)\right], \tag{4.23}$$

or

$$\lambda = \frac{1}{2}\tan^{-1}\left[\cos\frac{\beta^2}{8}\right]. \tag{4.24}$$

A similar result can be derived indirectly for the Ashkin–Teller parameter $K_4$. Wegner [20] has shown $K_4$ and $\lambda$ are related by duality according to

$$\tanh(2K_4) = \frac{-\tanh(2\lambda)}{1-\tanh(2\lambda)}. \tag{4.25}$$

Using (4.24) and the duality transformation $\beta \to 4\pi/\beta$, the relation between $\beta$ and $K_4$ is found to be

$$K_4 = \frac{1}{2}\tanh^{-1}\left[\frac{\cos(2\pi^2/\beta^2)}{\cos(2\pi^2/\beta^2)-1}\right]. \tag{4.26}$$

## Appendix A

This appendix is intended as a brief introduction to the two-dimensional free massless scalar field, and makes no pretentions to rigor. This model field theory is of interest here because it is the continuum limit of the Gaussian model (after the usual Wick rotation to Euclidean space). The Gaussian model is defined on a square lattice; at each lattice site $i$, there is a real variable $\phi_i$ which ranges between plus and minus infinity. The Euclidean action $A_G$ for the Gaussian model is a simple quadratic:

$$A_G = \sum_{\text{n.n.}} \frac{J^2}{2}[\phi_i - \phi_j]^2. \tag{A.1}$$

The parameter $J$ is a dimensionless coupling constant. The continuum limit of the

Gaussian model is the free massless scalar field theory, with (Minkowski-space) action

$$A = \int d^2x \frac{1}{2} (\partial\phi)^2. \tag{A.2}$$

As usual, the free field $\phi$ can be decomposed as

$$\phi(x) = \int \frac{d\vec{k}}{(4\pi\,|\vec{k}|^{1/2})} [a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^{+} e^{ik\cdot x}]. \tag{A.3}$$

There is another field, dual to $\phi$, which is also canonical and constructed from the same creation and annihilation operators as $\phi$. It is defined as

$$\tilde{\phi}(x) = \int \frac{d\vec{k}}{(4\pi\,|\vec{k}|)^{1/2}} \varepsilon(k)[a_{\vec{k}} e^{-ik\cdot x} + a_{k}^{+} e^{ik\cdot x}], \tag{A.4}$$

where $\varepsilon(k^1)$ is $\pm 1$, according to the sign of $k^1$. The fields $\phi$ and $\tilde{\phi}$ are related by

$$\partial_\mu \phi = \varepsilon_{\mu\nu} \partial^\nu \tilde{\phi}, \tag{A.5}$$

where $\varepsilon_{\mu\nu}$ is the usual totally antisymmetric tensor $[\varepsilon_{01} = -\varepsilon_{10} = +1]$.

The fields $\phi$ and $\tilde{\phi}$ are not themselves observables. It $\phi$ were observable, it would be the Goldstone boson associated with the spontaneous breaking of the continuous symmetry $\phi \to \phi + \lambda$. This is forbidden in two-dimensional field theories [27], so the symmetry must be manifest. However, exponentials of $\phi$ and $\tilde{\phi}$, i.e., coherent states, are observables. It is convenient to define operators

$$\theta_\alpha(x) = \exp i\alpha\phi(x), \tag{A.6a}$$

$$\tilde{\theta}_\beta(y) = \exp i\beta\tilde{\phi}(y). \tag{A.6b}$$

Normal-ordering will always be understood.

There is a simple master formula for the vacuum expectation value of products of $\theta_\alpha$'s and $\tilde{\theta}_\beta$'s. It is

$$\begin{aligned}\langle 0|\prod_{i=1}^{m} \theta_{\alpha_i}(x_i) \prod_{k=1}^{n} \tilde{\theta}_{\beta_k}(yk)\,|0\rangle = \delta_{A,0}\delta_{B,0} \exp \Big\{ &\sum_{i<j} -\alpha_i\alpha_j\Delta_+(x_i - x_j) \\ &+ \sum_{k<l} -\beta_k\beta_l\Delta_+(y_k - y_l) + \sum_{i,k} -\alpha_i\beta_k\tilde{\Delta}_+(x_i - y_k)\Big\}.\end{aligned} \tag{A.7}$$

$A$ and $B$ are defined by

$$A = \sum_{i=1}^{m} \alpha_i, \tag{A.8a}$$

$$B = \sum_{k=1}^{n} \beta_k. \tag{A.8b}$$

The Kronecker deltas, $\delta_{A,0}$ and $\delta_{B,0}$, insure that the continuous symmetries of the theory are manifest. The two-point functions $\Delta_+$ and $\tilde{\Delta}_+$ are given by

$$\Delta_+(x) = -\frac{1}{4\pi} \ln[-\mu^2 x^2 + i\varepsilon x^0], \tag{A.9a}$$

$$\tilde{\Delta}_+(x) = -\frac{1}{4\pi} \ln\left[\frac{x^0 - x' - i\varepsilon}{x^0 + x' - i\varepsilon}\right]. \tag{A.9b}$$

In (A.9a), $\mu$ is an arbitrary mass parameter; its value can be changed by a multiplicative renormalization of $\theta_\alpha$ and $\tilde{\theta}_\beta$.

Of particular interest are the spin-wave and vortex operators defined by

$$S_m = \exp \frac{im}{J} \phi, \tag{A.10a}$$

$$V_n = \exp i2\pi nJ\tilde{\phi}. \tag{A.10b}$$

After continuation to Euclidean space, the general spin-wave and vortex correlation function is given by

$$\left\langle \prod_{i=1}^{m} S_{m_i}(x_i) \prod_{k=1}^{n} V_{n_k}(y_k) \right\rangle = \delta_{\Sigma_i m_i,0} \delta_{\Sigma_k n_k,0} \exp \left\{ \sum_{i<j} \frac{m_i m_j}{2\pi J^2} \ln \mu |x_i - x_j| + \sum_{k<l} n_k n_l 2\pi J^2 \ln \mu |y_k - y_l| + \sum_{i,k} - i m_i n_k \theta(x_i - y_k) \right\}. \tag{A.11}$$

The arctangent function $\theta$ is defined (up to a multiple of $2\pi$) as

$$\theta(x) = \tan^{-1}(x^1/x^2). \tag{A.12}$$

The spin-wave operator $S_m$ is defined with a factor of $J^{-1}$ so that it has the same large-distance behavior as the Gaussian model observable $\exp[im\phi_j]$. The vortex operator, on the other hand, is defined with a factor of $2\pi J$ so that mixed vortex–spin-wave correlation functions do not change under a rotation of $2\pi$. These vortex operators are the continuum limit of similar objects found in the Gaussian model.

The spin-wave and vortex operators possess an important duality property. As can easily be seen from Eq. (A.11), correlation functions are invariant under the duality transformation

$$V_n \leftrightarrow S_n, \tag{A.13}$$

$$2\pi J^2 \leftrightarrow 1/2\pi J^2. \tag{A.14}$$

## Appendix B

Consider a model field theory with Euclidean Lagrangian density

$$L_1 = \frac{1}{2}(\nabla\phi)^2 + \frac{2h_p}{a^2}\cos\left(\frac{p}{J}\phi\right). \tag{B.1}$$

The generating functional $Z$ for this model can be written as a path integral:

$$Z = \int [d\phi]\, e^{-\int d^2x L_1}. \tag{B.2}$$

$Z$ can be thought of as the partition function for the continuum limit of a Gaussian model to which a $p$-fold symmetry-breaking field has been applied. It is easy to expand $Z$ as a power series in $h_p$.

$$\begin{aligned} Z &= 1 + \sum_{n=1}^{\infty} \left(\frac{h_p}{a^2}\right)^n \frac{1}{n!} \int d^2x_1 \cdots d^2x_n \left\langle \prod_{j=1}^{n} S_p(x_j) + S_p^+(x_j)] \right\rangle \\ &= 1 + 2h_p^2 \int \frac{d^2x_1}{a^2}\frac{d^2x_2}{a^2} \exp\left[\frac{-p^2}{2\pi J^2} \ln \pi\mu\, |x_1 - x_2|\right] + \cdots \end{aligned} \tag{B.3}$$

A similar expansion can be performed for the model described by Eq. (1.1), using the formula (A.11) of Appendix A.

From this expansion follow the duality relations described in the text. Order by order in $y$ and $h_p$, one works with correlation functions of the Gaussian model, which obey the duality relations (A.13) and (A.14). It follows easily that the model described by Eq. (1.1) is invariant under the duality transformation

$$2\pi J^2 \leftrightarrow p^2/2\pi J^2, \tag{B.4}$$

$$y \leftrightarrow h_p. \tag{B.5}$$

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## References

1. J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Phys. Rev. B* **16** (1977), 1217.
2. L. P. Kadanoff, *J. Phys. A* **11** (1978), 1399.
3. S. Elitzur, R. Pearson, and J. Shigemitsu, *Phys. Rev. D* **19** (1979), 3698.
4. L. P. Kadanoff, *Ann. Phys. (N.Y.)* **120** (1979), 39.

5. L. P. Kadanoff and A. C. Brown, *Ann. Phys. (N.Y.)* **121** (1979), 318.
6. H. A. Kramers and G. H. Wannier, *Phys. Rev.* **60** (1941), 252.
7. L. P. Kadanoff and H. Ceva, *Phys. Rev. B* **3** (1971), 3918.
8. T. D. Schultz, D. C. Mattis, and E. H. Lieb, *Rev. Mod. Phys.* **36** (1964), 856.
9. R. A. Ferrell, *J. Stat. Phys.* **8** (1973), 265.
10. G. t'Hooft, *Nucl. Phys. B* **138** (1978), 1.
11. S. Mandelstam, *Phys. Rev. D* **11** (1975), 3026.
12. J. A. Swieca, *Fortschr. Phys.* **25** (1977), 303.
13. S. Coleman, *Phys. Rev. D* **11** (1975), 2088.
14. $\gamma^0 = \sigma_x$, $\gamma^1 = i\sigma_y$, and $\gamma_5 = \gamma^0\gamma^1 = -\sigma_z$.
15. M. Sato, T. Miwa, and M. Jimbo, *Proc. Japan. Acad. Sect. A* **53** (1977), 147.
16. R. J. Baxter, *Phys. Rev. Lett.* **26** (1971), 832.
17. L. P. Kadanoff and F. J. Wegner, *Phys. Rev. B* **11** (1971), 3989.
18. J. Ashkin and E. Teller, *Phys. Rev.* **64** (1942), 178.
19. C. Fan and F. Wu, *Phys. Rev. B* **2** (1970), 723.
20. F. J. Wegner, *J. Phys. C* **5** (1972), 181.
21. B. Schroer and T. Truong, *Nucl. Phys. B* **136** (1978), 475.
22. A. Luther and I. Peschel, *Phys. Rev. B* **12** (1975), 3908.
23. B. Berg, *Ann. Phys. (N.Y.)* **110** (1978), 329.
24. B. Sutherland, *J. Math. Phys.* **11** (1970), 3183.
25. M. Lüscher, *Nucl. Phys. B* **117** (1976), 475.
26. J. Villain, *J. Phys. (Paris)* **36** (1975), 581.
27. S. Coleman, *Comm. Math. Phys.* **31** (1973), 259.
28. A lattice discussion of the commutator is given by E. Fradkin and L. P. Kadanoff, University of Illinois preprint.